Context

VishwaNath Govind Rao

An easy rule to solve the equation $a^n+b^n=c^n$ for n

[If a, b,be positive and $(b/a)^{|n|} <10^{10}$, $(b>a)$, then the equation $a^n+b^n=c^n$ may be solved for n by repeated application of the formula $n= [\log4][\log\{c^2-(a-b)^2\}-\log(ab)]^{-1}$]

Shri VishwaNath Govind Rao has tried his level best to solve the equation $a^n+b^n=c^n$. I will feel extremely grateful to all the learned Mathematicians in case if they provide proper guide line to the solution.

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After reading the solution of the equation $a^n+b^n=c^n$ given by shri VishwaNath Govind Rao, I have come to the conclusion that he is successful in his attempt.He should be congratulated for such type of attempt. I hope that one, who will read the solution, will be fully satisfied.

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An easy rule to solve the equation $a^n+b^n=c^n$ for n

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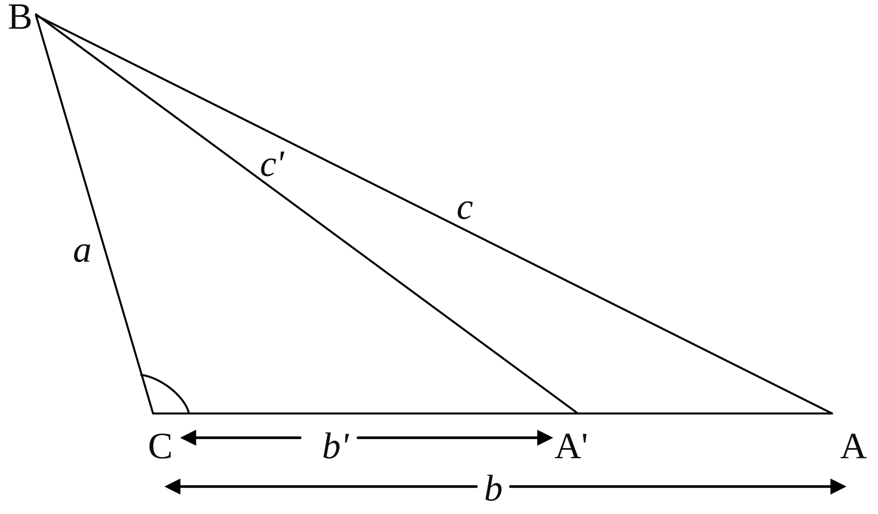
To solve the equation $a^n+b^n=c^n$ for n, we construct a ΔABC,



where AB=c, BC=a, CA=b.

We take a point A' on CA such that BC=CA'.

Suppose CA'=b' and A'B=c'. And let

$$a^{n_1'}+b^{n_1'}=c^{n_1'} \text{ ....(i)},$$

$$a^{n_1}+b'^{n_1}=c'^{n_1} \text{ ....(ii)}.$$

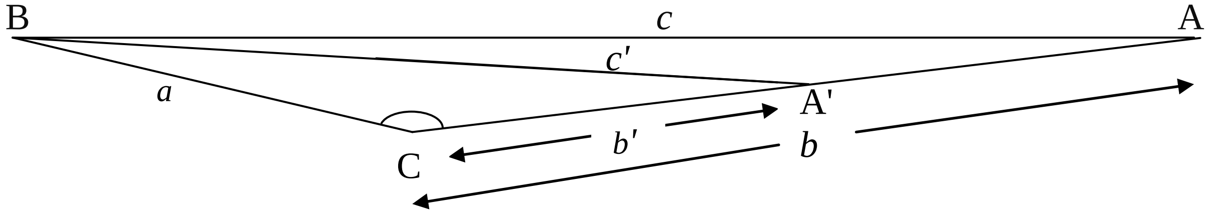
Then the principle of solving the equation $a^n+b^n=c^n$ is based upon two facts

(A) $n_1=n_1'$ approximately, when $\angle C \neq \pi, \frac{\pi}{2}$.

[this approximation may, however, be very rough.]

(B) $n_1=n_1'$ exactly, when $\angle C = \pi, \frac{\pi}{2}$.

[as $\angle C \to \pi, \frac{\pi}{2}$, n_1 coincides with n_1']



For, when $\angle C \to \pi$, the eqn $a^n+b^n=c^n$ tends to the identity $x^1+y^1=(x+y)^1$.

i.e., $\angle C \to \pi \Leftrightarrow n \to 1$.

Now considering the eqn (ii)

i.e., $a^{n_1} + b'^{n_1} = c'^{n_1}$

or $a^{n_1} + a^{n_1} = c'^{n_1}$ [a=b']

or $2a^{n_1} = c'^{n_1}$

or $(2a^{n_1})^2 = (c'^{n_1})^2$ [squaring both the sides]

or $4(a^2)^{n_1} = (c'^2)^{n_1}$

or $\log 4 + n_1 \log a^2 = n_1 \log c'^2$ [taking logs on both the sides]

$= n_1 \log (a^2 + b'^2 - 2ab' \cos C)$

$= n_1 \log (2a^2 - 2a^2 \cos C)$ [a=b']

$= n_1 \log \{2a^2(1 - \cos C)\}$

$$= n_1 \log \left\{ 2a^2 \left(1 - \frac{a^2 + b^2 - c^2}{2ab} \right) \right\}$$

$$= n_1 \log \left\{ 2a^2 \left(\frac{c^2 - (a^2 + b^2 - 2ab)}{2ab} \right) \right\}$$

$$= n_1 \log \left\{ a^2 \frac{c^2 - (a-b)^2}{ab} \right\}$$

or $\log 4$ $$= n_1 \log \left\{ a^2 \frac{c^2 - (a-b)^2}{ab} \right\} - n_1 \log a^2$$

$$= n_1 \log \left\{ a^2 \frac{c^2 - (a-b)^2}{ab} \frac{1}{a^2} \right\}$$

$$= n_1 \log \frac{c^2 - (a-b)^2}{ab}$$

or n_1 $$= \frac{\log 4}{\log \frac{c^2 - (a-b)^2}{ab}} \qquad \text{….(iii)}$$

=the approximate value of n_1'. [fact (A)]

i.e., Corresponding to n_1' we find an n_1, approximately equal to n_1'.We shall use this result in finding $n_2, n_3, \ldots$ for the approximate values of $n_2', n_3' \ldots$.

Now considering the eqn (i).

i.e., $a^{n_1'} + b^{n_1'} = c^{n_1'}$

Replacing n_1' by n_1 and multiplying the new index n_1 by n_2' to balance the equation.

We get, $\left(a^{n_1}\right)^{n_2'} + \left(b^{n_1}\right)^{n_2'} = \left(c^{n_1}\right)^{n_2'}$ .... (iv)

Applying the result (iii) upon this equation, we find

$$n_2 = \frac{\log 4}{\log \dfrac{\left(c^{n_1}\right)^2 - \left(a^{n_1} - b^{n_1}\right)^2}{a^{n_1} b^{n_1}}}.$$

= the approximate value of n_2' [fact (A)]

Now considering the eqn (iv).

Replacing n_2' by n_2 and multiplying the new index $n_1 n_2$ by n_3' to balance the equation.

We get, $\left(a^{n_1 n_2}\right)^{n_3'} + \left(b^{n_1 n_2}\right)^{n_3'} = \left(c^{n_1 n_2}\right)^{n_3'}$.

Applying the result (iii) upon this equation, we find

$$n_3 = \frac{\log 4}{\log \dfrac{\left(c^{n_1 n_2}\right)^2 - \left(a^{n_1 n_2} - b^{n_1 n_2}\right)^2}{a^{n_1 n_2} b^{n_1 n_2}}}.$$

= the approximate value of n_3' [fact (A)]

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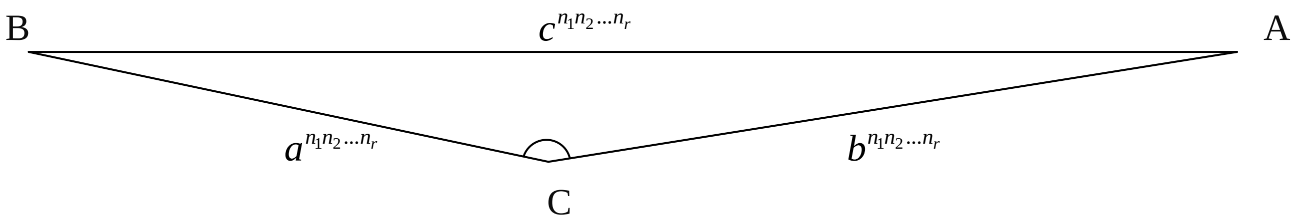
We get, $(a^{n_1 n_2 \ldots n_r})^{n_{r+1}'} + (b^{n_1 n_2 \ldots n_r})^{n_{r+1}'} = (c^{n_1 n_2 \ldots n_r})^{n_{r+1}'}$ ....(v).

Applying the result(iii) again upon this equation, we find

$$n_{r+1} = \frac{\log 4}{\log \dfrac{\left(c^{n_1 n_2 \ldots n_r}\right)^2 - \left(a^{n_1 n_2 \ldots n_r} - b^{n_1 n_2 \ldots n_r}\right)^2}{a^{n_1 n_2 \ldots n_r} b^{n_1 n_2 \ldots n_r}}}.$$

= the approximate value of n_{r+1}' [fact (A)].



Now if $n_{r+1} \to 1$, it means $\angle C \to \pi$. Then we must have,

$$n_{r+1} = n'_{r+1} = 1. \qquad \text{[fact(B)]}$$

Replacing n'_{r+1} by 1 in eqn (v),

We get, $$a^{n_1n_2...n_r} + b^{n_1n_2...n_r} = c^{n_1n_2...n_r}.$$

Comparing this eqn to the given eqn $a^n+b^n=c^n$,we have,

$n_1n_2n_{3.....}n_r \to n$

or $n_1n_2n_{3.....}n_r\, n_{r+1} \to n$. [for $n_{r+1} \to 1$]

N.B.

Let $b>a$

(i) If $c \leq |(a-b)|, <a, <b$, we should replace a,b,c, by their reciprocals and n by $(-n)$.(pl. see example 4 on page 12)

(ii) a,b, should be positive, or, the term $(a^{N_r} - b^{N_r})$ may be imaginary.

(iii) Fortunately, it is easier to make $n_{r+1} \to 1$ for the lower values of $(b/a)^{|n|}$. (Pl. see example 2 on page -8)

(iv) If $(b/a)^{|n|} > 10$ then in the process of finding the values of n, if these values fluctuate, we should begin to take the mean {one or more than one times according to the value of $(b/a)^{|n|}$} of the two successive values of n,i.e., our intention is to make $n_{r+1 \to 1}$, and the process of making so is arbitrary. (Pl.see example 3 on page -9)

(v) We may compare this process, to the process, invented by Newton to find the approximate solution of equations.
(Pl. see the "Text-Book On Differential Calculus" by Gorakh Prasad,D.Sc., Eleventh Edition-1968,page-81)

(vi) As we proceed, better value of n is obtained. i.e.,$n_1n_2...n_{r+1}$ is better than $n_1n_2...n_r$.

Example-1

Let us solve the eqn

$7^n+8^n=6^n$

Let $N_1, N_2, \ldots\ldots, N_r$ are the successive(approxi.) values of n obtained in solving the above equation.(i.e., Let $N_r=n_1n_2n_{3\ldots\ldots}n_r$)

$$\text{Then } N_1 = n_1 = \frac{\log 4}{\log \frac{6^2-(7-8)^2}{7\times 8}} = -2.94953969, \qquad [n_1=-2.94953969]$$

$$N_2=n_1n_2=N_1\frac{\log 4}{\log \frac{6^{2N_1}-(7^{N_1}-8^{N_1})^2}{56^{N_1}}} = \underline{-3.24}627234, \qquad [n_2=1.100603037]$$

$$N_3=n_1n_2n_3=N_2\frac{\log 4}{\log \frac{6^{2N_2}-(7^{N_2}-8^{N_2})^2}{56^{N_2}}} = \underline{-3.2428}35749, \qquad [n_3=0.998941373]$$

$$N_4= n_1n_2n_3n_4=N_3\frac{\log 4}{\log \frac{6^{2N_3}-(7^{N_3}-8^{N_3})^2}{56^{N_3}}} = \underline{-3.2428}80485, \qquad [n_4=1.000013795]$$

$$N_5= n_1n_2n_3n_4n_5=N_4\frac{\log 4}{\log \frac{6^{2N_4}-(7^{N_4}-8^{N_4})^2}{56^{N_4}}} = \underline{-3.2428799}04, \qquad [n_5=0.99999982]$$

$$N_6= n_1n_2n_3n_4n_5n_6=N_5\frac{\log 4}{\log \frac{6^{2N_5}-(7^{N_5}-8^{N_5})^2}{56^{N_5}}}, \qquad [n_6=1.000000002]$$

$$=N_5\times 1.000000002=-\underline{3.24287991}$$

Here 1.000000002 is nearly equal to 1,

$$\therefore n = -3.24287991\ldots$$

Example-2

Let us solve the eqn

$7^n+50^n=5448^n$

We have been well familiar with $n_1 n_2 n_3 \ldots n_r$, so we omit this and we shall write only N_r for $n_1 n_2 n_{3\ldots} n_r$.

Then $N_1=\dfrac{\log 4}{\log \dfrac{5448^2-(7-50)^2}{7\times 50}}=.\underline{12}2161866,$ $[n_1=0.122161866]$

$N_2=N_1\dfrac{\log 4}{\log \dfrac{5448^{2N_1}-\left(7^{N_1}-50^{N_1}\right)^2}{350^{N_1}}}=.\underline{1234}61217,$ $[n_2=1.010636306]$

$N_3=N_2\dfrac{\log 4}{\log \dfrac{5448^{2N_2}-\left(7^{N_2}-50^{N_2}\right)^2}{350^{N_2}}}=.\underline{1234559}53,$ $[n_3=0.999957363]$

$N_4=N_3\dfrac{\log 4}{\log \dfrac{5448^{2N_3}-\left(7^{N_3}-50^{N_3}\right)^2}{350^{N_3}}}=\underline{.123455975}\ ,$ $[n_4=1.000000178]$

$N_5=N_4\dfrac{\log 4}{\log \dfrac{5448^{2N_4}-\left(7^{N_4}-50^{N_4}\right)^2}{350^{N_4}}}\ ,$ $[n_5=0.999999999]$

$=N_4\times.999999999=.123455975$

Here .999999999 is nearly equal to 1,

$\therefore n=.123455975\ldots$

Example-3

Let us solve the eqn

$3^n+7^n=7.000000015^n$

This miscellaneous example will clear, what we have written in N.B. (iv) on page-6.

$$\text{Hear } N_1=\frac{\log 4}{\log\frac{7.000000015^2-(3-7)^2}{3\times 7}}=3.067$$

$$N_2=N_1\frac{\log 4}{\log\frac{7.000000015^{2N_1}-(3^{N_1}-7^{N_1})^2}{21^{N_1}}}=6.488$$

$$N_3=N_2\frac{\log 4}{\log\frac{7.000000015^{2N_2}-(3^{N_2}-7^{N_2})^2}{21^{N_2}}}=13.014$$

$$N_4=N_3\frac{\log 4}{\log\frac{7.000000015^{2N_3}-(3^{N_3}-7^{N_3})^2}{21^{N_3}}}=25.964$$

$$N_5=N_4\frac{\log 4}{\log\frac{7.000000015^{2N_4}-(3^{N_4}-7^{N_4})^2}{21^{N_4}}}=6.006$$

There is fluctuation,

$$\therefore N_6=\frac{1}{2}(N_4+N_5)=15.985$$

$\left[\left(\frac{7}{3}\right)^{N_6} = \frac{733010}{6digits}, \frac{6}{3} = 2\right.$, We should take the mean (two times) of the two successive values of *n*.$\left.\right]$

$$\text{Now } N_7 = N_6 \frac{\log 4}{\log \frac{7.000000015^{2N_6} - (3^{N_6} - 7^{N_6})^2}{21^{N6}}} = 30.823$$

$$\therefore N_8 = \frac{1}{2}\left\{\frac{1}{2}\left(N_7 + N_6\right) + N_6\right\} = 19.694$$

$\left[\left(\frac{7}{3}\right)^{N_8} = \frac{17657227}{8digits}, \frac{8}{3} = 3\right.$(near an integer). We should take the mean (three times) of the two successive values of *n*.$\left.\right]$

$$\text{Now } N_9 = N_8 \frac{\log 4}{\log \frac{7.000000015^{2N_8} - (3^{N_8} - 7^{N_8})^2}{21^{N_8}}} = 21.839$$

$$\therefore N_{10} = \frac{1}{2}\left[\frac{1}{2}\left\{\frac{1}{2}\left(N_9 + N_8\right) + N_8\right\} + N_8\right] = 19.9622$$

$$N_{11} = N_{10} \frac{\log 4}{\log \frac{7.000000015^{2N_{10}} - (3^{N_{10}} - 7^{N_{10}})^2}{21^{N_{10}}}} = \underline{20}.3496$$

$$N_{12} = \frac{1}{2}\left[\frac{1}{2}\left\{\frac{1}{2}\left(N_{11} + N_{10}\right) + N_{10}\right\} + N_{10}\right] = \underline{20}.0106$$

$$N_{13}=N_{12}\frac{\log 4}{\log\dfrac{7.000000015^{2N_{12}}-(3^{N_{12}}-7^{N_{12}})^2}{21^{N_{12}}}}=\underline{20.0}838$$

$$N_{14}=\frac{1}{2}\left[\frac{1}{2}\left\{\frac{1}{2}\left(N_{13}+N_{12}\right)+N_{12}\right\}+N_{12}\right]=\underline{20.0}1975$$

$$N_{15}=N_{14}\frac{\log 4}{\log\dfrac{7.000000015^{2N_{14}}-(3^{N_{14}}-7^{N_{14}})^2}{21^{N_{14}}}}=\underline{20.0}3361$$

$$N_{16}=\frac{1}{2}\left[\frac{1}{2}\left\{\frac{1}{2}\left(N_{15}+N_{14}\right)+N_{14}\right\}+N_{14}\right]=\underline{20.02}148$$

$$N_{17}=N_{16}\frac{\log 4}{\log\dfrac{7.000000015^{2N_{16}}-(3^{N_{16}}-7^{N_{16}})^2}{21^{N_{16}}}}$$

$$=N_{16}\times 1.000087458=\underline{20.02}$$

Here 1.000087 is nearly equal to 1,

$\therefore$ we may conclude that n=20.02….

If we proceed further and further to find $N_{18},\ldots,N_{30}$,we find the more accurate value of n as n=20.0216615

Example-4

Let us solve the eqn

$$17^n+25^n=7^n$$

Here 7<|17-25|,<17,<25. So,we replace 17,25,7 by their reciprocals and *n* by (-*n*) to get

$$\left(17^{-1}\right)^{-n}+\left(25^{-1}\right)^{-n}=\left(7^{-1}\right)^{-n}$$

Solving this eqn for (-*n*), we get

$$N_1=\frac{\log 4}{\log\frac{\left(7^{-1}\right)^2-(17^{-1}-25^{-1})^2}{(17\times 25)^{-1}}}=\underline{0.64}6968374$$

$$N_2=N_1\frac{\log 4}{\log\frac{7^{-2N_1}-(17^{-N_1}-25^{-N_1})^2}{(17\times 25)^{-N_1}}}=\underline{0.6489}61768$$

$$N_3=N_2\frac{\log 4}{\log\frac{7^{-2N_2}-(17^{-N_2}-25^{-N_2})^2}{(17\times 25)^{-N_2}}}=\underline{0.648953}113$$

$$N_4=N_3\frac{\log 4}{\log\frac{7^{-2N_3}-(17^{-N_3}-25^{-N_3})^2}{(17\times 25)^{-N_3}}}$$

$$=N_3\times 1.000000058=\underline{0.64895315}$$

Here 1.000000058 is nearly equal to 1

$$\therefore -n=0.64895315$$

i.e., n=-0.64895315

Example-5

The population growth of three cities A, B&C are 5%, 10% and 15% respectively per year. If their populations are the same at this time, to determine the time in years when the sum of the populations of cities A&B will be equal to the population of city C alone.

Solution

Let their population at this time be P and the required time be n.Then

$$P\left(1+\frac{5}{100}\right)^n + P\left(1+\frac{10}{100}\right)^n = P\left(1+\frac{15}{100}\right)^n$$

or $1.05^n+1.10^n=1.15^n$

Then $$N_1=\frac{\log 4}{\log \frac{1.15^2-(1.05-1.10)^2}{1.05\times 1.10}}=\underline{10}\mathbf{.}38$$

$$N_2=N_1\frac{\log 4}{\log \frac{1.15^{2N_1}-\left(1.05^{N_1}-1.10^{N_1}\right)^2}{1.155^{N_1}}}=\underline{10.6}936$$

$$N_3=N_2\frac{\log 4}{\log \frac{1.15^{2N_2}-\left(1.05^{N_2}-1.10^{N_2}\right)^2}{1.155^{N_2}}}=\underline{10.688}$$

$$N_4=N_3\frac{\log 4}{\log \frac{1.15^{2N_3}-\left(1.05^{N_3}-1.10^{N_3}\right)^2}{1.155^{N_3}}}$$

$$=N_3\times 1.000049=\underline{10.68852}$$

Here 1.000049 is nearly equal to 1,

$\therefore$ The required time will be 10.68852 years (approxi.). If we proceed further and further to find N_5,N_6,we find the more accurate value of n as n=10.68852122…

Example-6

The increase in diameters of three plants A, B&C are 30%, 40%, and 45% respectively per year.If their weights vary as the cubes of their diameters, to determine the time in years when the sum of the weights of the plants A&B will be equal to the weight of the plant C alone, assuming that the diameter of each plant be the same at this time (i.e., when t=0).

Solution

Let when t=0, the diameter of each plant be D and the required time be n, then the wts of the plants A,B and C will be $K\left[D\left(1+\frac{30}{100}\right)^n\right]^3$, $K\left[D\left(1+\frac{40}{100}\right)^n\right]^3$ and $K\left[D\left(1+\frac{45}{100}\right)^n\right]^3$ respectively, where K is proportionality constant.

Now according to the above condition.

$$K\left[D\left(1+\frac{30}{100}\right)^n\right]^3 + K\left[D\left(1+\frac{40}{100}\right)^n\right]^3 = K\left[D\left(1+\frac{45}{100}\right)^n\right]^3$$

$$\text{i.e., } 1.3^{3n}+1.4^{3n}=1.45^{3n}$$

If we apply the previous procedure to find $N_1, N_2, \ldots, N_8$, we find

$$3n=10.66242126$$

$$\text{i.e., } n=3.55414042\ldots\text{years.}$$